# Trace distance

- Classically:

  Statistical distance

- Want: something like SD, but for q. states.

---

Given two density ops $\rho, \sigma$ how well can we distinguish?

Informally:

$$TD(\rho,\sigma) = \max_{Adv} \left| \Pr[Adv(\rho) \to yes] - \Pr[Adv(\sigma) \to yes] \right|$$

---

<u>Case 1</u>: $\rho, \sigma$ classical

i.e. $\rho$ corresponds to

$$\{ |i\rangle \, @ \, \underbrace{p_i} \} \rightarrow P$$

$\sigma$ corr. to $\{|i\rangle @ r_i\} \rightsquigarrow \mathcal{R}$

Want:

$$TD(\rho,\sigma) = SD(\mathcal{P},\mathcal{R})$$

$$= \frac{1}{2}\sum_i |p_i - r_i|$$

$$\rho = \sum p_i |i\rangle\langle i| = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_N \end{pmatrix}$$

$$\sigma = \begin{pmatrix} r_1 & & & \\ & r_2 & & \\ & & \ddots & \\ & & & r_N \end{pmatrix}$$

$$|\rho-\sigma| = \begin{pmatrix} |p_1-r_1| & & \\ & \ddots & \\ & & |p_N-r_N| \end{pmatrix}$$

$$\frac{1}{2}\operatorname{tr}|\rho-\sigma| = \frac{1}{2}\sum_i |p_i - r_i|$$

$$TD(\rho,\sigma) := \frac{1}{2}\operatorname{tr}|\rho-\sigma|$$

If $A$ is diag: $|A|$ is elem-wise abs value

What if $\rho - \sigma$ is not diag?

<u>Need</u>: a def of $|A|$
even if $A$ not diag.

Def: $|A| := \sqrt{A^\dagger A}$ $\left(\underset{\mathbb{C}}{|a|} = \sqrt{\bar{a} a}\right)$

Def: $\sqrt{B}$ is the
(for $B \geq 0$) unique matrix $C \geq 0$
s.t. $C^\dagger C = B$

$\left(\sqrt{\underset{\geq 0}{b}} = \text{unique } c \geq 0 \text{ s.t. } c^2 = b\right)$

- $|A|$ is elem-wise abs. value for diag $A$
- If $A = UBU^\dagger$ (unitary $U$), then $|A| = U|B|U^\dagger$

Now we can def. TD:

$$\boxed{TD(\rho,\sigma) = \frac{1}{2} \operatorname{tr} |\rho - \sigma|}$$

$$TD(\rho,\sigma) = \frac{1}{2} \operatorname{tr} |\overbrace{\rho - \sigma}^{=:A}|$$

$$= \frac{1}{2} \operatorname{tr} |A| = \frac{1}{2} \operatorname{tr} |\underbrace{UDU^{+}}_{\text{diag. of } A}|$$

$$= \frac{1}{2} \operatorname{tr} U |D| U^{+}$$

$$= \frac{1}{2} \operatorname{tr} |D|$$

↳ has eigenvals of $A$ on diag.

$$= \frac{1}{2} \sum_i |e_i|$$ where $e_i$ are eigenvalues of $A$

$\Rightarrow$ This is a recipe for comp. trace distance.

# Properties of TD

- $TD(\rho,\sigma) = SD(\mathcal{P}, R)$
  if $\rho, \sigma$ corr. to class. distr. $\mathcal{P}, R$

| | |
|---|---|
| SD is a metric | TD is a metric ($\geq 0$, $=0$ iff $\rho=\sigma$, triangle ineq.) |
| $SD \leq 1$ | $TD \leq 1$ |
| For any (poss. prob.) function $F$: $SD(F(X), F(Y)) \leq SD(X,Y)$ | For any q. operation $\mathcal{E}$: $TD(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq TD(\rho,\sigma)$ |
| If $Z$ is indep of $X,Y$: $SD((X,Z), (Y,Z)) = SD(X,Y)$ | For density ops $\rho, \sigma, \tau$: $TD(\rho\otimes\tau, \sigma\otimes\tau) = TD(\rho,\sigma)$ |

Does :

$$TD(\rho,\sigma) = \max_{Adv} \Big| \Pr[Adv(\rho) \to yes] - \Pr[Adv(\rho) \to no] \Big|$$

hold?

Can we find optimal distinguisher?

Say $TD(\rho,\sigma) = \varepsilon$.

Assume $\rho-\sigma$ is diagonal, diag elems are decreasing

$$\rho-\sigma = \begin{pmatrix} d_1 & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & d_N \end{pmatrix} \quad \begin{matrix} \} > 0 \\ \} = 0 \\ \} < 0 \end{matrix}$$

$$d_1 \geq d_2 \; \ldots \; \geq d_N$$

$$P := \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \; \begin{matrix} \} \text{"}\geq 0 \text{ block"} \\ \\ \\ \end{matrix} , \quad M = \{P, 1-P\}$$

$\Pr[M(\rho) \to yes] = \mathrm{tr}\, P\rho$ | $\Pr[M(\sigma) \to yes] = \mathrm{tr}\, P\sigma$

$$\left| \Pr[M(\rho) \to yes] - \Pr[M(\sigma) \to yes] \right|$$

$$= \left| \mathrm{tr}\, P(\rho - \sigma) \right|$$

$$= \left| \sum_{d_i > 0} d_i \right| = \sum_{d_i > 0} d_i$$

$$\varepsilon = TD(\rho, \sigma) = \frac{1}{2} \mathrm{tr}\, |\rho - \sigma|$$

$$= \frac{1}{2} \sum_i |d_i| = \frac{1}{2}\left( \sum_{d_i > 0} d_i \underbrace{- \sum_{d_i < 0} d_i}_{= \sum_{d_i > 0} d_i} \right)$$

$$= \frac{1}{2}\left( 2 \sum_{d_i > 0} d_i \right) = \sum_{d_i > 0} d_i$$

$\boxed{\text{Note: } \mathrm{tr}\,\rho - \sigma = \mathrm{tr}\,\rho - \mathrm{tr}\,\sigma = 1 - 1 = 0 \Rightarrow \sum d_i = 0}$

$$\Rightarrow \left| \Pr[M(\rho) \to yes] - \Pr[M(\sigma) \to yes] \right| = TD(\rho, \sigma)$$